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Non-commutativity as a measure of inequivalent quantization

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Abstract
We show that the strength of non-commutativity could play a role in determining the boundary condition of a physical problem. As a toy model, we consider the inverse-square problem in non-commutative space. The scale invariance of the system is explicitly broken by the scale of non-commutativity Θ. The effective problem in non-commutative space is analyzed. It is shown that despite the presence of a higher singular potential coming from the leading term of the expansion of the potential to first order in Θ, it can have a self-adjoint extension. The boundary conditions are obtained, which belong to a 1-parameter family and are related to the strength of non-commutativity.

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(Some figures in this article are in colour only in the electronic version)

The study of non-commutative spacetime [1, 2] is a fascinating subject. The expectation that the spacetime could be non-commutative at small length scale has further accelerated research in this direction. Due to the non-commutativity of coordinates of a plane (x, y), there exists an uncertainty relation,

$$\Delta x \Delta y \sim \Theta, \tag{1}$$

where Θ is the non-commutativity parameter. Non-commutativity of a charged particle can arise due to the nontrivial nature of spacetime at small length scale or it may arise if the magnetic field, subjected perpendicular to the plane, is strong enough. However, the idea of non-commutativity of spacetime is quite old. Non-commutativity was used in the work of Snyder in 1947 [3], although it did not get much attention at the time. In quantum theory, non-commutativity is a key object, for example coordinate x and its conjugate p are non-commutative:

$$\Delta x \Delta p \sim \hbar. \tag{2}$$

Even the generalized momenta P_i in the magnetic field background B do not commute:

$$\Delta P_1 \Delta P_2 \sim B. \tag{3}$$

The coordinates of a plane behave as canonical conjugate pairs and therefore do not commute in the presence of a strong magnetic field perpendicular to the plane.

The strength of non-commutativity, Θ, may have an intrinsic origin in spacetime or it may have origin in an external magnetic field as stated previously. However, the length scale, Θ, introduced in the problem due to the non-commutativity can be exploited to heal the ultraviolet divergence of the problem under study. In a recent paper [4], we investigated the inverse-square problem, $H = \boldsymbol{p}^2 + \alpha \boldsymbol{r}^{-2}$, in non-commutative space in order to show how the length scale Θ can be successfully used to regularize the problem. Since the inverse-square problem does not possess any dimensional parameter to start with, it is a scale-invariant problem. It can be understood from the transformation $\mathbf{r} \to \varepsilon\mathbf{r}$ and $t \to \varepsilon^2 t$. The parameter ε is the scaling factor. One can check that the classical action corresponding to the Hamiltonian H is invariant under this transformation. Note that the Hamiltonian H transforms as $H \to (1/\varepsilon^2)H$. The Lagrangian L associated with the system also transforms in the same way, $L \to (1/\varepsilon^2)L$. It is now obvious that the action, $\mathcal{A} = \int \mathrm{d}t L$, will be scale invariant under the transformation $\mathbf{r} \to \varepsilon\mathbf{r}$ and $t \to \varepsilon^2 t$. In quantum mechanics, it has the following consequences. Let ϕ be an eigenstate of the Hamiltonian H with the eigenvalue E, i.e., $H\phi = E\phi$, then $\phi_\varepsilon = \phi\left(\frac{\mathbf{r}}{\varepsilon}\right)$ will also be an eigenstate of the same H but with energy $\frac{E}{\varepsilon^2}$. The ground state, therefore, has no lower bound, implying that it does not have any bound state. It is, however, known from some physical problems, for example binding of an electron in a polar molecule [5], the near horizon states of a black hole [6] and other [7–10] that inverse-square potential can bind particles. The theoretical interpretation of this binding can be obtained in terms of nontrivial quantization, which can be obtained by the von Neumann method of self-adjoint extensions.

However, once the inverse-square problem is considered in a non-commutative plane, it looses its scale symmetry property due to the presence of dimensional parameter Θ. To first order in the parameter, Θ, the potential $V = \alpha/r^2$ in the non-commutative plane becomes more singular, but then it belongs to an interesting class of interaction $V_\mu = g/r^\mu, \mu > 2$, studied in [11]. The interesting feature of the potential V_μ is that it possesses a localized state at the threshold of energy $E = 0$. The state which has zero eigenvalue is usually considered as a transition point from bound states to scattering states. But due to the nontrivial asymptotic nature of the potential of the type V_μ, they can form bound states [12], even at $E = 0$. Apart from scale symmetry, the inverse-square problem has even larger symmetry, formed by three generators: the Hamiltonian H, the dilatation generator $\mathcal{D}$ and the conformal generator K. It is called the $SO(2,1)$ algebra: $[\mathcal{D}, H] = -\mathrm{i}\hbar H, [\mathcal{D}, K] = \mathrm{i}\hbar K, [H, K] = 2\mathrm{i}\hbar\mathcal{D}$ [13, 14]. We have shown that with the introduction of non-commutativity the $so(2,1)$ symmetry of the system is broken explicitly and, however, in the commutative limit the exact $so(2,1)$ symmetry is restored.

In the present paper, we extend our discussion of [4] further and obtain a generic boundary condition for the zero-energy localized state. The paper is organized in the following fashion: first, we consider the inverse-square interaction on a plane and discuss briefly how it changes when the coordinates of the plane become non-commutative. Second, we consider the non-commutative Hamiltonian obtained to first order in the non-commutativity parameter Θ. The possible bound state spectrum is discussed in terms of generic boundary conditions. Finally, we conclude with some discussion.

We now consider a particle, interacting with a potential $V = \alpha/r^2$ on a non-commutative plane with the algebra

$$[\hat{x}_1, \hat{x}_2] = 2\mathrm{i}\Theta, \qquad [\hat{p}_1, \hat{p}_2] = 0, \qquad [\hat{x}_i, \hat{p}_j] = \mathrm{i}\hbar\delta_{ij}. \tag{4}$$

However, the commutative limit $\Theta \to 0$ takes it to the standard algebra:

$$[x_1, x_2] = 0, \qquad [p_1, p_2] = 0, \qquad [x_i, p_j] = \mathrm{i}\hbar\delta_{ij}. \tag{5}$$

It is useful to get a representation of the non-commutative coordinates $(\hat{x}_i, \hat{p}_i)$ in terms of the coordinates (x_i, p_i). We choose a representation,

$$\begin{aligned} &\hat{x}_1 = x_1 - \Theta p_2, \qquad \hat{x}_2 = x_2 + \Theta p_1, \\ &\hat{p}_1 = p_1, \qquad \hat{p}_2 = p_2, \end{aligned} \tag{6}$$

for our purpose, but other representations are also possible. The Hamiltonian on the non-commutative plane

$$H_{\mathrm{NC}} = \hat{p}_1{}^2 + \hat{p}_2{}^2 + \alpha/\hat{r}^2, \tag{7}$$

to first order in the non-commutative parameter Θ can be written as

$$H_{\mathrm{NC}} = p_1{}^2 + p_2{}^2 + \alpha/r^2 + 2\alpha\Theta(x_1p_2 - x_2p_1)/r^4. \tag{8}$$

The presence of the potential $2\alpha\Theta(x_1p_2 - x_2p_1)/r^4$ breaks the scale invariance. We solve the eigenvalue problem,

$$H_{\mathrm{NC}}\psi_{\mathrm{NC}} = E_{\mathrm{NC}}\psi_{\mathrm{NC}}, \tag{9}$$

for $E_{\mathrm{NC}} = 0$ and found a bound state with angular momentum m for $\xi = \sqrt{\alpha + m^2} > 1$ [4]. For large values of the non-commutative parameter, Θ, it is also possible to get the expectation values of the Hamiltonian. Since the zero-energy Schrödinger equation is exactly solvable, it is possible to ask what is the most general boundary condition in this case. To be explicit, we consider an eigenvalue problem of the form

$$\widehat{H_{\mathrm{NC}}}\psi_{\mathrm{NC}} \equiv -\frac{r^4}{\alpha m}\left(p_1^2 + p_2^2 + \alpha/r^2\right)\psi_{\mathrm{NC}} = 2\Theta\psi_{\mathrm{NC}}. \tag{10}$$

Note that the dimensional parameter 2Θ has been considered as the eigenvalue for our problem. All square-integrable solutions for different values of the parameter Θ correspond to the $E_{\mathrm{NC}} = 0$ degenerate states. Even for complex values of the parameter Θ if the solution ψ_{NC} is square integrable then it corresponds to the bound state with $E_{\mathrm{NC}} = 0$. Since our assumption in (4) is that the parameter Θ is real, we will restrict the parameter space to be real. It can be done if we can ensure that $\widehat{H_{\mathrm{NC}}}$ is self-adjoint. From now onward the symmetric operator $\widehat{H_{\mathrm{NC}}}$ will be investigated and a suitable boundary condition will be found out, which will make the operator self-adjoint.

Imposing a well-defined boundary condition is important for getting a physical solution. In this paper, we exploit von Neumann's method to analyze $\widehat{H_{\mathrm{NC}}}$. Before actually making any symmetric extensions for the operator $\widehat{H_{\mathrm{NC}}}$, a brief discussion about the von Neumann's method is necessary here. Consider any symmetric operator, say $\mathcal{B}$, which is for the moment taken to be unbounded. It is possible to define a domain $D(\mathcal{B})$ under which the operator $\mathcal{B}$ is symmetric. One can also obtain the adjoint operator, $\mathcal{B}^*$, corresponding to the operator $\mathcal{B}$. From the symmetric condition $\int_0^\infty \phi^*(r)\mathcal{B}\chi(r)\mathrm{d}r = \int_0^\infty (\mathcal{B}^*\phi(r))^*\chi(r)\mathrm{d}r, \forall\chi(r) \in D(\mathcal{B})$ we can obtain the domain, $D(\mathcal{B}^*)$. The operator $\mathcal{B}$ would be self-adjoint if the two domains are same, i.e., $D(\mathcal{B}) = D(\mathcal{B}^*)$. In terms of the deficiency indices $n_\pm$ [15], one can have alternative definition of self-adjointness. The deficiency indices $n_\pm$ are the dimension of the kernel, $\mathrm{Ker}(\mathrm{i} \pm \mathcal{B}^*)$. If $n_\pm = 0$, then the operator $\mathcal{B}$ is essentially self-adjoint. If $n_+ = n_- = n \neq 0$, then $\mathcal{B}$ is not self-adjoint but admits a self-adjoint extension. It can be characterized by n^2 parameters. Different values of the parameters give rise to different physics. For, $n_+ \neq n_-$, operator $\mathcal{B}$ does not have any self-adjoint extensions.

The operator $\widehat{H_{\mathrm{NC}}}$, which we are analyzing in this work, acts on the functions defined on a Hilbert space of square-integrable functions with the domain $\mathcal{L}^2[R^+, r\mathrm{d}r]$. Since the solution of the problem (10) has a similarity with the inverse-square problem $H\psi = E\psi$ of [7], it would be helpful to look at the short distance and asymptotic behavior of both the solutions. One can check that the solutions have an inverse relation to each other of the form

$$\lim_{r\to 0} \psi_{NC} \equiv \lim_{r\to\infty} \psi, \qquad \lim_{r\to\infty} \psi_{NC} \equiv \lim_{r\to 0} \psi. \tag{11}$$

Due to this inverse behavior of the eigenstate, we impose a nontrivial boundary condition for our problem at $r = \infty$. The operator $\widehat{H_{NC}}$ is essentially self-adjoint for $\xi^2 \geqslant 1$ which has been discussed in detail in [4]. Since any system is defined by a Hamiltonian and its corresponding domain, in our case $\widehat{H_{NC}}$ for $\xi^2 \geqslant 1$ acts on the domain

$$\mathcal{D}_0 = \{\psi \in \mathcal{L}^2(r\mathrm{d}r), \qquad \psi(\infty) = \psi'(\infty) = 0\}. \tag{12}$$

Note the difference that the same condition (12) was imposed for the inverse-square problem [7] but at $r \to 0$. Let us now investigate the operator in the interval $\xi \in (-1, 1)$. In this region, $\widehat{H_{NC}}$ is not essentially self-adjoint and, therefore, we need to make self-adjoint extensions of the original domain, so that the Hamiltonian becomes self-adjoint. We discuss the case $\xi \neq 0$ first, and then consider the case $\xi = 0$ separately. The deficiency indices are $\langle 1, 1\rangle$ for $\xi \in (-1, 1)$. Since the number of deficiency space solutions is the same for both types, there exists a self-adjoint extension, characterized by a parameter, Σ. The domain under which $\widehat{H_{NC}}$ would be self-adjoint is given by

$$\mathcal{D}_\Sigma = \{\mathcal{D}_0 + \psi_+ + e^{\mathrm{i}\Sigma}\psi_-\}. \tag{13}$$

The explicit form of the deficiency space solutions $\psi_\pm$ are given by

$$\psi_+ = H_\xi\left(\frac{\sqrt{\alpha m}}{r}\mathrm{e}^{-\mathrm{i}\pi/4}\right), \tag{14}$$

$$\psi_- = H_\xi\left(\frac{\sqrt{\alpha m}}{r}\mathrm{e}^{+\mathrm{i}\pi/4}\right), \tag{15}$$

where H_ξ is the modified Bessel function [16]. The behavior of any function, belonging to the domain $\mathcal{D}_\Sigma$, near $r \to \infty$ can be found from the behavior of $\psi_+ + \mathrm{e}^{\mathrm{i}\Sigma}\psi_-$ at the asymptotic limit. Because the domain $\mathcal{D}_0$ goes to zero at $r \to \infty$, it does not contribute to the domain at $r \to \infty$. The asymptotic behavior of the domain is of the form

$$\lim_{r\to\infty}\left(\psi_+ + \mathrm{e}^{\mathrm{i}\Sigma}\psi_-\right) \simeq \mathcal{A}_+ (2r)^{-\xi} + \mathcal{A}_- (2r)^{\xi}, \tag{16}$$

where, $\mathcal{A}_\pm = -\frac{(\alpha m)^{\pm\xi/2}\pi\mathrm{i}}{\sin(\pi\xi)}\frac{\cos(\frac{\Sigma}{2} \pm \frac{\pi\xi}{4})}{\Gamma(1\pm\xi)}$. The solution of (10) has to be matched with (16) to get the relation of the non-commutativity parameter Θ with the self-adjoint extension parameter Σ. We see that there is exactly one bound state with the non-commutativity, 2Θ, and eigenfunction, ψ_{NC}, being of the form

$$2\Theta = \frac{1}{\alpha m}\sqrt[\xi]{\frac{\cos\frac{1}{4}(2\Sigma + \xi\pi)}{\cos\frac{1}{4}(2\Sigma - \xi\pi)}}, \tag{17}$$

$$\psi_{NC} = \exp(\mathrm{i}m\phi)H_\xi\left(\frac{\sqrt{-2\Theta\alpha m}}{r}\right). \tag{18}$$

In figure 1 the behavior of the parameter 2Θ as a function of the self-adjoint extension parameter Σ has been shown for three different values of the coupling constant α and for the fixed value of the angular momentum quantum number m. Now let us come to the case for

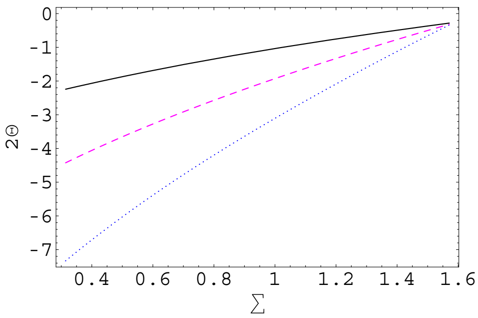

Figure 1. A plot of the non-commutativity parameter 2Θ as a function of the self-adjoint extension parameter Σ for $m = 1$. It corresponds to equation (17). The blue (dotted) curve corresponds to $\alpha = -1/10$, the pink (dashed) curve corresponds to $\alpha = -1/6$ and the black (full) curve corresponds to $\alpha = -1/3$.

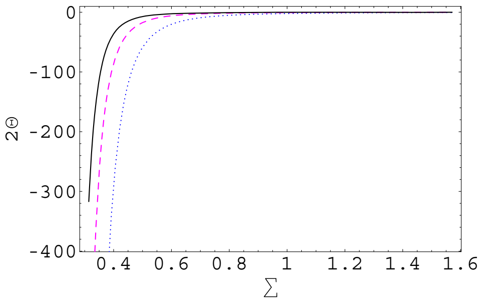

Figure 2. A plot of equation (19). The blue (dotted) graph corresponds to $\alpha = -4$ and $m = 2$. The pink (dashed) graph corresponds to $\alpha = -9$ and $m = 3$. The black (full) graph corresponds to $\alpha = -16$ and $m = 4$.

$\xi = 0$, which can be handled similarly. The non-commutativity parameter corresponding to the bound state and the corresponding eigenstate are given by

$$2\Theta = \frac{1}{\alpha m} \exp\left(\frac{\pi}{2} \cot \frac{\Sigma}{2}\right), \tag{19}$$

$$\psi_{\mathrm{NC}} = \exp(\mathrm{i}m\phi) K_0 \left(\frac{\sqrt{-2\Theta\alpha m}}{r}\right), \tag{20}$$

respectively, where K_0 [16] is the modified Bessel function. In figure 2, the parameter 2Θ of (19) has been plotted as a function of the self-adjoint extension parameter Σ for three sets of values of the pair α and m. Note that since the non-commutativity parameter Θ comes out as an eigenvalue of an effective Hamiltonian it leads to quantization of Θ. In [17], the non-commutativity parameter is also obtained to be quantized, although there the reason was due to Dirac quantization condition of the monopole charge.

So far we have considered only the term which is first order in the non-commutativity parameter Θ. However, it is possible to consider higher-order terms. Although the differential equation may not be exactly solvable, the question of possible self-adjoint extensions can be answered with the help of Wyle's limit point-limit circle criterion [15]. It says that a potential

$V(r)$ of a Hamiltonian $H = -\frac{d^2}{dr^2} + V(r)$ is in the limit circle case at $r = 0$ or $r = \infty$ if for some E and therefore for all E all solutions of $H\psi = E\psi$ are square integrable at $r = 0$ or $r = \infty$. If $V(r)$ is not in the limit circle case then it is in the limit point case. Any operator H can be then classified as

(i) If $V(r)$ is in the limit circle case at both ends then $\langle n_+, n_-\rangle = \langle 2, 2\rangle$ and H admits a 4-parameter family of self-adjoint extensions.
(ii) If $V(r)$ is in the limit circle case at one end and the limit point case at the other end then $\langle n_+, n_-\rangle = \langle 1, 1\rangle$ and H admits a 1-parameter family of self-adjoint extensions.
(iii) If $V(r)$ is in the limit point case at both ends then $\langle n_+, n_-\rangle = \langle 0, 0\rangle$ and H is essentially self-adjoint.

In our case, the Hamiltonian with terms higher order in non-commutativity is of the form

$$H_{\rm NC} = p_1^2 + p_2^2 + \frac{\alpha}{r^2} + \frac{2\alpha\Theta L_z}{r^4} + \Theta^2\frac{\alpha}{r^2}\left(\frac{4L_z^2}{r^4} - p_1^2 - p_2^2\right) + \mathcal{O}(\Theta^3). \tag{21}$$

In the asymptotic limit $r \to \infty$, all potential terms can be neglected compared to $\frac{\alpha}{r^2}$ term. The two independent solutions then become $\sim r^{\pm\xi}$, both of which are square integrable for $\xi \in (-1, 1)$, which means that $H_{\rm NC}$ is in the limit circle case at $r \to \infty$ for all order in the non-commutativity parameter Θ. Since one end is in the limit circle case, the Hamiltonian $H_{\rm NC}$ belongs to either category 1 or 2 above, which implies that it has self-adjoint extensions.

We adopted a representation (6) where the Hamiltonian takes a simple symmetric form, and therefore the eigenvalue equation is exactly solvable. However, one can consider different representations but the result would be the same. For example, consider

$$\hat{x}_1 = x_1 - 2\Theta p_2, \qquad \hat{x}_2 = x_2, \qquad \hat{p}_1 = p_1, \qquad \hat{p}_2 = p_2, \tag{22}$$

or

$$\hat{x}_1 = x_1, \qquad \hat{x}_2 = x_2 + 2\Theta p_1, \qquad \hat{p}_1 = p_1, \qquad \hat{p}_2 = p_2. \tag{23}$$

Both the representations are compatible with the algebra (4). One can check taking the representation (22) that to first order in non-commutativity $H_{\rm NC}$ is

$$H_{\rm NC} = p_1^2 + p_2^2 + \frac{\alpha}{r^2} + \frac{4\alpha\Theta x_1 p_2}{r^4}. \tag{24}$$

Note that (7) is the symmetrized version of the operator (24). Similarly, taking the other representation (23) one will get a potential term $-\frac{4\alpha\Theta x_2 p_1}{r^4}$, which can be symmetrized to the desired potential term of (8).

Although we used a specific representation in our calculation which enables us to map the non-commutative quantum-mechanical problem to the plane where coordinates commute, it can be shown that this approach is equivalent to working in the non-commutative plane itself [18]. In the non-commutative plane, two functions $\psi(x)$ and $\chi(x)$ do not commute due to a product known as Moyal product or $\star$-product of the form

$$\psi(x) \star \chi(x) = e^{i\Theta^{ij}\partial_i^{(1)}\partial_j^{(2)}}\psi(y)\chi(z)|_{y=z=x}. \tag{25}$$

The Schrödinger equation in this plane can be written by replacing all ordinary product by the $\star$-product as

$$i\frac{\partial\psi(x;t)}{\partial t} = \left(\frac{p^2}{2m} + V(x)\right) \star \psi(x;t). \tag{26}$$

Note that the kinetic term is not affected by the $\star$-product, only the potential term is affected by the $\star$-product. One can use Bopp's shift to get the effect of non-commutativity. For detailed discussion, see [18] and the references therein. The coordinates x_i of the potential $V(x)$ are shifted by $x_i - \Theta\epsilon_{ij}p_j$, where ϵ_{ij} is the antisymmetric tensor taking values $\epsilon_{12} = -\epsilon_{21} = 1$. The Schrödinger equation (26) then is equivalent to

$$\mathrm{i}\frac{\partial\psi(x;t)}{\partial t} = \left(\frac{p^2}{2m} + V(x_i - \Theta\epsilon_{ij}p_j)\right)\psi(x;t), \qquad (27)$$

where now the coordinates are commutative among themselves. Note that the representation (6) we used in our analysis is nothing but Bopp's shift.

Finally, to first order in non-commutativity, Θ, the inverse-square problem has been discussed as a toy model to illustrate the connection of the boundary conditions with the strength of non-commutativity. The exact solvability of the $E_{\mathrm{NC}} = 0$ eigenstate has been exploited to get a generic boundary condition by making a suitable self-adjoint extensions for the problem. We treated the non-commutativity Θ as the eigenvalue and obtained a generic boundary conditions under which the spectra are restricted to the subspace of real axis.

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